



THE TOPOLOGY OF BALLS AND GROMOV HYPERBOLICITY OF RIEMANN SURFACES

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§0. ABSTRACT

We prove that every ball in any non-exceptional Riemann surface with radius less or equal than $\frac{1}{2} \log 3$ is either simply or doubly connected. We use this theorem in order to study the hyperbolicity in the Gromov sense of Riemann surfaces. The results clarify the role of punctures and funnels of a Riemann surface in its hyperbolicity.

§1. INTRODUCTION

A good way to understand the important connections between graphs and Potential Theory on Riemannian manifolds (see e.g. [ARY], [CFPR], [FR2], [HS], [K1], [K2], [K3], [R1], [R2], [So]) is to study the Gromov hyperbolic spaces. This approach allows to establish a general setting to work simultaneously with graphs and manifolds, in the context of metric spaces. Besides, the idea of Gromov hyperbolicity grasps the essence of negatively curved spaces, and has been successfully used in the theory of groups (see e.g. [GH] and the references therein).

Although there exist some interesting examples of hyperbolic spaces (see the examples after Definition 2.1), the literature gives no good guide about how to determine whether or not a space is hyperbolic. This limitation can be somehow got round, since the theory allows to obtain powerful results about non-hyperbolic spaces which have hyperbolic universal coverings. As topological “obstacles” may prevent a space from being hyperbolic, the possibility of studying its universal covering instead, which is always free of obstacles, implies a substantial simplification, and sometimes let us extract important information about the space itself (see [P]).

However, as was stated above, the characterization of hyperbolic spaces remains open. Recently, some interesting results about the hyperbolicity of Euclidean bounded domains with their quasihyperbolic metric have made significant progress in this direction (see [BHK] and the references therein).

Originally, we were interested in studying when non-exceptional Riemann surfaces equipped with its Poincaré metric were Gromov hyperbolic. However, we have proved two theorems on hyperbolicity for general metric spaces, which are interesting by themselves (see Section 2) and have important consequences for Riemann surfaces (see Section 3). Although one should expect Gromov hyperbolicity in non-exceptional Riemann surfaces due to its constant curvature -1 , this turns out to be untrue in general, since topological obstacles can impede it: for instance, the two-dimensional jungle-gym (a \mathbf{Z}^2 -covering of a torus with genus two) is not hyperbolic. Let us recall that in the case of modulated plane domains, quasihyperbolic metric and Poincaré metric are equivalent.

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We prove in [RT2] that there is no inclusion relationship between hyperbolic Riemann surfaces and the usual classes of Riemann surfaces, such as O_G , O_{HP} , O_{HB} , O_{HD} , surfaces with hyperbolic isoperimetric inequality, or the complements of these classes (even in the case of plane domains). This fact makes the study of hyperbolic Riemann surfaces more complicated and interesting. One can find results on hyperbolicity of Riemann surfaces in [RT1] and [RT2].

Here we present the outline of the main results. We refer to the next sections for the definitions and the precise statements of the theorems.

In Section 2 we obtain some lower bounds on the hyperbolicity constants of metric spaces, which will be useful in Section 3. In Section 3 we study the role of punctures and funnels of a Riemann surface in its hyperbolicity.

The main aim in this paper is obtaining global results on hyperbolicity from local information. That was the idea that led us to identify the punctures and funnels of a surface S^* with closed sets $\{E_n\}_n$ removed from an original surface S , in such a way that $S^* = S \setminus \cup_n E_n$.

Theorem 3.2 allows, in many cases, to forget punctures and funnels in order to study the hyperbolicity of a Riemann surface; this fact is a significant simplification in the topology of the surface, and therefore makes easier the problem. Besides, we have determined which are the relevant parameters in the hyperbolicity constant of S^* . If we consider just punctures, Theorem 3.4 gives a result with a statement much simpler than Theorem 3.2.

In order to prove Theorem 3.4 we need a universal result on the topology of balls in Riemann surfaces (see Theorem 3.1), which is interesting by itself: it says that every ball in any non-exceptional Riemann surface with radius less or equal than $\frac{1}{2} \log 3$ is either simply or doubly connected. Theorem 3.1 is a precise answer in our context to the question: when do geometric constraints imply topological ones? This is an attractive topic of research, as plenty of publications in first-rate quality journals show (see e.g. [Ch], [G], [GP], [GPW]).

As a consequence of Theorem 3.4, we have obtained interesting examples of stability of the hyperbolicity of Riemann surfaces (see Corollary 3.3).

We also prove a general criteria which guarantees that many surfaces are not hyperbolic (see Theorem 3.3).

It is a remarkable fact that almost every constant appearing in the results of this paper depends just on a small number of parameters. This is a common place in the theory of hyperbolic spaces (see e.g. theorems A, B and C, and Lemma B) and is also typical of surfaces with curvature -1 (see e.g. the Collar Lemma in [R] and [S], and Theorem 3.1).

Notations. We denote by X or X_n geodesic metric spaces. By d_X , L_X and B_X we shall denote, respectively, the distance, the length and the balls in the metric of X .

We denote by S or S_i non-exceptional Riemann surfaces. We assume that the metric defined on these surfaces is the Poincaré metric.

Finally, we denote by k_i positive constants which can assume different values in different theorems.

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§2. RESULTS IN METRIC SPACES

In our study of hyperbolic Gromov spaces we use the notations of [GH]. We give now the basic facts about these spaces. We refer to [GH] for more background and further results.

Definition 2.1. Let us fix a point w in a metric space (X, d) . We define the *Gromov product* of $x, y \in X$ with respect to the point w as

$$(x|y)_w := \frac{1}{2} (d(x, w) + d(y, w) - d(x, y)) \geq 0.$$

We say that the metric space (X, d) is δ -hyperbolic ($\delta \geq 0$) if

$$(x|z)_w \geq \min \{ (x|y)_w, (y|z)_w \} - \delta,$$

for every $x, y, z, w \in X$. We say that X is *hyperbolic* (in the Gromov sense) if the value of δ is not important.

It is convenient to remark that this definition of hyperbolicity is not universally accepted, since sometimes the word hyperbolic refers to negative curvature or to the existence of Green's function. However, in this paper we only use the word *hyperbolic* in the sense of Definition 2.1.

Examples: (1) Every bounded metric space X is $(\text{diam } X)$ -hyperbolic (see e.g. [GH, p. 29]).

(2) Every complete simply connected Riemannian manifold with sectional curvature which is bounded from above by $-k$, with $k > 0$, is hyperbolic (see e.g. [GH, p. 52]).

(3) Every tree with edges of arbitrary length is 0-hyperbolic (see e.g. [GH, p. 29]).

Definition 2.2. If $\gamma : [a, b] \rightarrow X$ is a continuous curve in a metric space (X, d) , we can define the length of γ as

$$L(\gamma) := \sup \left\{ \sum_{i=1}^n d(\gamma(t_{i-1}), \gamma(t_i)) : a = t_0 < t_1 < \dots < t_n = b \right\}.$$

We say that γ is a *geodesic* if it is an isometry, i.e. $L(\gamma|_{[t,s]}) = d(\gamma(t), \gamma(s)) = |t - s|$ for every $s, t \in [a, b]$. We say that X is a *geodesic metric space* if for every $x, y \in X$ there exists a geodesic joining x and y ; we denote by $[x, y]$ any of such geodesics (since we do not require uniqueness of geodesics, this notation is ambiguous, but it is convenient). It is clear that every geodesic metric space is path-connected.

Definition 2.3. If X is a geodesic metric space and $J = \{J_1, J_2, \dots, J_n\}$, with $J_j \subseteq X$, we say that J is δ -thin if for every $x \in J_i$ we have that $d(x, \cup_{j \neq i} J_j) \leq \delta$. If $x_1, x_2, x_3 \in X$, a *geodesic triangle* $T = \{x_1, x_2, x_3\}$ is the union of three geodesics $[x_1, x_2]$, $[x_2, x_3]$ and $[x_3, x_1]$. The space X is δ -thin (or satisfies the *Rips condition* with constant δ) if every geodesic triangle in X is δ -thin.

A basic result is that hyperbolicity is equivalent to Rips condition:

Theorem A. ([GH, p. 41]) *Let us consider a geodesic metric space X .*

- (1) *If X is δ -hyperbolic, then it is 4δ -thin.*
- (2) *If X is δ -thin, then it is 4δ -hyperbolic.*

We present now the class of maps which play the main role in the theory.

Definition 2.4. A function between two metric spaces $f : X \longrightarrow Y$ is a *quasiisometry* if there are constants $a \geq 1$, $b \geq 0$ with

$$\frac{1}{a} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b, \quad \text{for every } x_1, x_2 \in X.$$

A such function is called an (a, b) -*quasiisometry*. An (a, b) -*quasigeodesic* in X is an (a, b) -quasiisometry between an interval of \mathbf{R} and X . An (a, b) -*quasigeodesic segment* in X is an (a, b) -quasiisometry between a compact interval of \mathbf{R} and X .

Quasiisometries are important since they are the maps which preserve hyperbolicity (see e.g. [GH, p. 88]). Notice that a quasiisometry can be discontinuous.

Along this paper we will work with topological subspaces of a geodesic metric space X . There is a natural way to define a distance in these spaces:

Definition 2.5. If X_0 is a path-connected subset of a geodesic metric space (X, d) , then we associate to it the *restricted distance*

$$d_{X_0}(x, y) := d_X|_{X_0}(x, y) := \inf \{L(\gamma) : \gamma \subset X_0 \text{ is a continuous curve joining } x \text{ and } y\} \geq d_X(x, y).$$

If X_0 is not path-connected, we also use this definition if x and y belong to the same path-connected component of X_0 ; if x and y belong to distinct path-connected components of X_0 , we define $d_{X_0}(x, y) := \infty$.

The following result will be useful in order to decide that a geodesic metric space is not hyperbolic (see Theorem 3.3).

Theorem 2.1. *Let us consider a geodesic metric space X , and $X_1, X_2 \subset X$ two geodesic metric spaces such that $X_1 \cap X_2 = \eta_1 \cup \eta_2$, with η_i compact sets, $\text{diam}_{X_i}(\eta_j) \leq c_1$ for any $i, j = 1, 2$, and $d_X(\eta_1, \eta_2) \geq c_2$. Then there exists a geodesic triangle $T = \{a, b, c\}$ in X and $x \in [a, b]$ with $d_X(x, [a, c] \cup [b, c]) \geq c_2/2 - c_1$.*

Remark. We will see in the proof of the theorem that the conclusion is also true if we change the hypothesis “ η_i are compact sets”, by “there exist geodesics γ_i in X_i joining η_1 and η_2 , with $L_X(\gamma_i) = d_{X_i}(\eta_1, \eta_2)$ ”.

Proof. Without loss of generality, we can assume that $c_2 \geq 2c_1$, since if this was not so, the conclusion is clear. Since η_1, η_2 are compact sets, we have that there exist geodesics γ_i in X_i joining η_1 and η_2 , with $L_X(\gamma_i) = d_{X_i}(\eta_1, \eta_2)$.

Without loss of generality, we can assume that $L_X(\gamma_1) \leq L_X(\gamma_2)$; then it is not difficult to see that γ_1 is also a geodesic in X : it is clear that a geodesic γ in X such that $L_X(\gamma) = d_X(\eta_1, \eta_2)$ must be completely contained in X_1 or in X_2 . If $\gamma_1 = [a, b]$, with $a \in \eta_1, b \in \eta_2$, let us consider a geodesic γ'_2 in X_2 joining a and b . Let us call c to the middle point of γ'_2 . We consider geodesics $[a, c], [b, c]$ in X , and the geodesic triangle T in X with these three geodesics joining a, b, c .

We see now that $[a, c]$ can not contain a geodesic connecting η_1 with η_2 in X_1 : If $[a, c]$ contains such geodesic, we call it g ; then $L_X(g) \geq d_X(\eta_1, \eta_2) \geq c_2$. If $L_X(\gamma'_2) = 2r$, then we have that

$$d_X(c, \eta_1 \cup \eta_2) = d_{X_2}(c, \eta_1 \cup \eta_2) \geq \min \{d_{X_2}(c, a) - \text{diam}_{X_2}(\eta_1), d_{X_2}(c, b) - \text{diam}_{X_2}(\eta_2)\} \geq r - c_1.$$

Consequently $r = L_X(\gamma'_2)/2 = L_X([a, c]) \geq d_X(c, \eta_1 \cup \eta_2) + L_X(g) \geq r - c_1 + c_2$, which is a contradiction with $c_2 \geq 2c_1$. Hence, $L_X([a, c] \cap X_1) \leq c_1$ and $d_X(p, \eta_1) \leq c_1$ for every $p \in [a, c] \cap X_1$. A similar result holds for $[b, c]$.

Consequently, if x is the middle point of γ_1 , then

$$d_X(x, [a, c] \cup [b, c]) \geq d_X(\eta_1, \eta_2)/2 - c_1 \geq c_2/2 - c_1. \quad \square$$

In the applications we usually know $d_{X_2}(\eta_1, \eta_2)$, but we do not have any lower bound of $d_X(\eta_1, \eta_2)$ at all. We can obtain a similar result to Theorem 2.1 with just a bound of $d_{X_2}(\eta_1, \eta_2)$, if we work with quasigeodesic triangles.

Definition 2.6. Let us consider three quasigeodesics J_1 starting in x_1 and finishing in x_2 , J_2 starting in x_2 and finishing in x_3 , J_3 starting in x_3 and finishing in x_1 , in a metric space. We say that $T = \{J_1, J_2, J_3\}$ is an (a, b) -*quasigeodesic triangle* if J_1, J_2, J_3 are (a, b) -quasigeodesics.

We need the following elementary result.

Lemma A. ([PRT, Lemma 3]) *Let us consider an (a, b) -quasigeodesic $q_1 : [\alpha, \beta] \rightarrow X$ and two continuous curves with arc-length parametrization $q_0 : [\alpha - d_1, \alpha] \rightarrow X$, $q_2 : [\beta, \beta + d_2] \rightarrow X$, verifying $q_0(\alpha) = q_1(\alpha)$ and $q_2(\beta) = q_1(\beta)$. Then the curve $q := q_0 \cup q_1 \cup q_2$ is an $(a, b + (1 + a^{-1})(d_1 + d_2))$ -quasigeodesic.*

The next result will be especially useful to decide that some spaces are not hyperbolic (see Corollary 2.1, theorems 3.2 and 3.3, and Lemma 3.1).

Theorem 2.2. *Let us consider a geodesic metric space X , and $X_1, X_2 \subset X$ two geodesic metric spaces such that $X_1 \cap X_2 = \eta_1 \cup \eta_2$, with η_i compact sets, $d_{X_2}(\eta_1, \eta_2) \geq c_2$ and $\text{diam}_{X_i}(\eta_j) \leq c_1$ for $i, j = 1, 2$. Then there exists a $(1, 2c_1)$ -quasigeodesic triangle $T = \{A, B, C\}$ in X and $x \in A$ with $d_X(x, B \cup C) \geq c_2/4$.*

Remark. The conclusion of Theorem 2.2 also holds if η_1 intersects η_2 (and even if $\eta_1 = \eta_2$); in this case we consider that η_1 and η_2 are disjoint sets in X_2 (they are identified if we paste X_1 and X_2 in order to obtain X).

Theorem 2.2 is a direct consequence of the following result.

Theorem 2.2'. *Let us consider a geodesic metric space X , and $X_1, X_2 \subset X$ two geodesic metric spaces such that $X_1 \cap X_2 = \eta_1 \cup \eta_2$, with η_i compact sets, $d_{X_1}(\eta_1, \eta_2) \leq d_{X_2}(\eta_1, \eta_2)$, $d_{X_2}(\eta_1, \eta_2) \geq c_2$ and $\text{diam}_{X_1}(\eta_j) \leq c_1$ for $j = 1, 2$. Then there exists a $(1, 2c_1)$ -quasigeodesic triangle $T = \{A, B, C\}$ in X and $x \in A$ with $d_X(x, B \cup C) \geq c_2/4$.*

Proof. Since η_1, η_2 are compact sets, we have that there exist geodesics γ_i in X_i joining η_1 and η_2 , with $L_X(\gamma_1) = d_{X_1}(\eta_1, \eta_2) \leq L_X(\gamma_2) = d_{X_2}(\eta_1, \eta_2)$. Without loss of generality, we can assume that $L_X(\gamma_2) = c_2$.

Let us denote by $a \in \eta_1$ and $b \in \eta_2$ the end points of γ_2 , and by c its middle point. We have that the two subcurves of γ_2 joining a with c , and b with c (both of length $c_2/2$), are geodesics in X : If there is some curve g in X joining a and c with $L_X(g) < c_2/2$, then there is some curve $g_0 \subseteq g$ joining c with η_1 or η_2 in X_2 with $L_X(g_0) < c_2/2$; consequently, we can construct a curve joining η_1 and η_2 in X_2 shorter than γ_2 . If there is some curve g in X joining b with c with $L_X(g) < c_2/2$, we have the same result.

Let us consider the triangle T in X with sides $[a, c], [b, c] \subset \gamma_2$ and γ_3 , where γ_3 is a continuous curve joining a with b in X_1 in the following way: γ_3 is the union of γ_1 and two geodesics in X_1 joining a with the end point of γ_1 belonging to η_1 , and b with the end point of γ_1 belonging to η_2 . By Lemma A we have that γ_3 is a $(1, 2c_1)$ -quasigeodesic, since $\text{diam}_{X_1}(\eta_j) \leq c_1$ for $j = 1, 2$. We define x as the middle point of $[a, c]$.

We only need to prove that $d_X(x, \gamma_3) = d_X(x, [b, c]) = c_2/4$:

Let us denote by p a point in γ_3 such that $d_X(x, \gamma_3) = d_X(x, p)$. Seeking a contradiction, suppose that there is some curve h in X joining x and p with $L_X(h) < c_2/4$. Then there is some curve $h_0 \subseteq h$ joining x with η_1 or η_2 in X_2 with $L_X(h_0) < c_2/4$; consequently, we can construct a curve joining η_1 and η_2 in X_2 shorter than γ_2 . Therefore $d_X(x, \gamma_3) \geq c_2/4$; since $d_X(x, a) = c_2/4$, we have that $d_X(x, \gamma_3) = c_2/4$.

Let us denote by q a point such that $d_X(x, [b, c]) = d_X(x, q)$. Seeking a contradiction, suppose that there is some curve r in X joining x and p with $L_X(r) < c_2/4$. If r intersects $\eta_1 \cup \eta_2$, we can use the same argument as in the previous case. If this was not so, the curve r is contained in X_2 ; since γ_2 is a geodesic in X_2 , we obtain $L_X(r) = d_X(x, p) = d_X(x, [c, b]) = d_X(x, c) = c_2/4$, which is a contradiction. Therefore $d_X(x, [c, b]) \geq c_2/4$; since $d_X(x, c) = c_2/4$, we have that $d_X(x, [c, b]) = c_2/4$. \square

In order to use Theorem 2.2 to guarantee that some spaces are not hyperbolic, we need the following elementary result.

Lemma B. ([PRT, Lemma 4]) *For each $\delta, b \geq 0$ and $a \geq 1$, there exists a constant $K = K(\delta, a, b)$ with the following property:*

If X is a δ -hyperbolic geodesic metric space and $T \subseteq X$ is an (a, b) -quasigeodesic triangle, then T is K -thin.

Corollary 2.1. *Let us consider a graph G which is a geodesic metric space, with a sequence of edges $\{e_n\}_n$ such that the graph $G \setminus e_n$ is a geodesic metric space for every n , and $\lim_{n \rightarrow \infty} L(e_n) = \infty$. Then G is not hyperbolic.*

We finish this section with two theorems which will be very useful in the proof of the main results of this paper. In order to state them, we need a definition.

Definition 2.7. We say that a geodesic metric space X has a *decomposition*, if there exists a family of geodesic metric spaces $\{X_n\}_{n \in \Lambda}$ with $X = \bigcup_{n \in \Lambda} X_n$ and $X_n \cap X_m = \bigcup_{i \in I_{nm}} \eta_{nm}^i$, where for

each $n \in \Lambda$, $\{\eta_{nm}^i\}_{m,i}$ are pairwise disjoint closed subsets of X_n ($\eta_{nm}^i = \emptyset$ is allowed); furthermore any geodesic segment in X meets at most a finite number of η_{nm}^i 's.

We say that X_n , with $n \in \Lambda$, is a (k_1, k_2, k_3) -tree-piece if it satisfies the following properties:

- (a) $\#I_{nm} \leq 1$ (then we can write $\eta_{nm}^i = \eta_{nm}$), $X \setminus \eta_{nm}$ is not connected for $m \neq n$ if $\#I_{nm} = 1$, and a, b are in different components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$
- (b) $\text{diam}_{X_n}(\eta_{nm}) \leq k_1$ for every $m \neq n$, and there exists $A_n \subseteq \Lambda$, such that $\text{diam}_{X_n}(\eta_{nm}) \leq k_2 d_{X_n}(\eta_{nm}, \eta_{nk})$ if $m \neq k$ and $m, k \in A_n$, and $\sum_{m \notin A_n} \text{diam}_{X_n}(\eta_{nm}) \leq k_3$.

We say that a geodesic metric space X has a *tree-decomposition* if it has a decomposition and there exist positive constants k_1, k_2, k_3 , such that every X_n , with $n \in \Lambda$, is a (k_1, k_2, k_3) -tree-piece.

We wish to emphasize that condition $\text{diam}_{X_n}(\eta_{nm}) \leq k_1$ is not very restrictive: if the space is “wide” at every point (in the sense of long injectivity radius, as in the case of simply connected spaces) or “narrow” at every point (as in the case of trees), it is easier to study its hyperbolicity; if we can found narrow parts (as η_{nm}) and wide parts, the problem is more difficult and interesting.

Remarks. 1. Obviously, condition (b) is required only for $\eta_{nm}, \eta_{nk} \neq \emptyset$.

2. The sets Λ and A_n do not need to be countable.

3. Condition (a) for every $n \in \Lambda$ guarantees that the graph $R = (V, E)$ constructed in the following way is a tree: $V = \cup_{n \in \Lambda} \{v_n\}$ and $[v_n, v_m] \in E$ if and only if $\eta_{nm} \neq \emptyset$.

4. If X is a Riemann surface and $\{X_n\}_{n \in \Lambda}$ are bordered Riemann surfaces and $\eta_{nm} \subset \partial X_n \cap \partial X_m$, condition “ a, b are in different components of $X \setminus \eta_{nm}$ for any $a \in X_n \setminus \eta_{nm}$, $b \in X_m \setminus \eta_{nm}$ ” in (a), is a consequence of “ $X \setminus \eta_{nm}$ is not connected”.

The following result can be applied to the study of the hyperbolicity of Riemann surfaces (see the proof of propositions 3.1 and 3.2). In [PRT] explicit expressions for the constants involved are supplied.

Theorem B. ([PRT, Theorem 1]) *Let us consider a tree-decomposition $\{X_n\}_{n \in \Lambda}$ of a geodesic metric space X . Then X is δ -hyperbolic if and only if there exists a constant k_4 such that X_n is k_4 -hyperbolic for every $n \in \Lambda$. Furthermore, if X is δ -hyperbolic, then k_4 only depends on δ, k_1, k_2 and k_3 ; if there exists k_4 , then δ only depends on k_1, k_2, k_3 and k_4 .*

Definition 2.8. We say that two geodesic metric spaces X and Y (in this order) have *comparable decompositions*, if there exist decompositions $\{X_n\}_{n \in \Lambda}$ of X and $\{Y_n\}_{n \in \Lambda}$ of Y , and constants k_i , with the following properties:

- (a) If $X_n \cap X_m = \cup_{i \in I_{nm}} \eta_{nm}^i$, then $Y_n \cap Y_m = \cup_{i \in I_{nm}} \sigma_{nm}^i$, and $\sigma_{nm}^i = \emptyset$ if and only if $\eta_{nm}^i = \emptyset$.
- (b) For any n, m, i , $\text{diam}_{X_n}(\eta_{nm}^i) \leq k_1$ and $\text{diam}_{Y_n}(\sigma_{nm}^i) \leq k_1$.
- (c) We can split Λ into $F \cup G$ and F into $F_1 \cup F_2$ with:
 - (c1) If $n \in G$, X_n is a (k_1, k_2, k_3) -tree-piece.
 - (c2) If $n \in F$, $\text{diam}_{X_n}(\eta_{nm}^i) \leq k_2 d_{X_n}(\eta_{nm}^i, \eta_{nk}^j)$ and $\text{diam}_{Y_n}(\sigma_{nm}^i) \leq k_2 d_{Y_n}(\sigma_{nm}^i, \sigma_{nk}^j)$ if $(m, i) \neq (k, j)$.
 - (c3) If $n \in F_1$, for each $\eta_{nm}^i \neq \eta_{nk}^j$, there exists a geodesic γ_{mnk}^{ij} in X_n , joining η_{nm}^i with η_{nk}^j , and a (k_4, b_{mnk}^{ij}) -quasiisometry $f_{mnk}^{ij} : \gamma_{mnk}^{ij} \longrightarrow h_{mnk}^{ij} \subseteq Y_n$, with h_{mnk}^{ij} starting in σ_{nm}^i and finishing in

σ_{nk}^j , and $\sum_{n \in F_1} \sum_{m,k,i,j} b_{mnk}^{ij} \leq k_5$, such that for any $x, y \in \cup_{m,k,i,j} \gamma_{mnk}^{ij}$, with corresponding points $x', y' \in \cup_{m,k,i,j} h_{mnk}^{ij}$, we have $k_4^{-1} d_{X_n}(x, y) - k_5 \leq d_{Y_n}(x', y')$.

(c4) If $n \in F_2$, there exists a $(k_4, 0)$ -quasiisometry $f_n : X_n \longrightarrow Y_n$, with $f_n(\eta_{nm}^i) \subseteq \sigma_{nm}^i$.

Remark. The hypothesis $\text{diam}_{X_n}(\eta_{nm}) \leq k_2 d_{X_n}(\eta_{nm}, \eta_{nk})$ holds if we have $d_{X_n}(\eta_{nm}, \eta_{nk}) \geq k_2'$, since $\text{diam}_{X_n}(\eta_{nm}) \leq k_1$.

The conditions that X_n must verify when n belongs to F_1, F_2 or G in Definition 2.8, is not arbitrary at all. In fact, what lies behind is an appropriate modelization for the situation which we will find in the proof of Theorem 3.2. The following theorem will be one of the important tools in the proof of Theorem 3.2. In [PRT] explicit expressions for the constants involved are supplied.

Theorem C. ([PRT, Theorem 2]) *Let us assume that two geodesic metric spaces X and Y have comparable decompositions. If Y is δ' -hyperbolic and there exists a constant k_6 such that X_n is k_6 -hyperbolic for every $n \in \Lambda \setminus F_2$, then X is δ -hyperbolic, with δ a constant which only depends on δ' and k_i .*

§3. RESULTS IN RIEMANN SURFACES

In this section we always work with the Poincaré metric; consequently, curvature is always -1 . In fact, many concepts appearing here (as punctures or funnels) only make sense with the Poincaré metric.

The intuition would say that negative curvature must imply hyperbolicity; in fact this is what happens when there are no topological “obstacles” (as in the case of the Poincaré disk \mathbf{D}) or if there is a finite number of them (see Proposition 3.2 in [RT1]). However, if there are infinitely many topological “obstacles”, the hyperbolicity can fail, as in the case of the two-dimensional jungle gym (a \mathbf{Z}^2 -covering of a torus with genus two).

The results in this section are useful since they not only provide many examples of hyperbolic Riemann surfaces, but also allow to establish criteria in order to decide whether a Riemann surface is hyperbolic or not.

Below we collect some definitions concerning to Riemann surfaces which will be referred to afterwards.

An *open non-exceptional* Riemann surface (or a non-exceptional Riemann surface without boundary) S is a Riemann surface whose universal covering space is the unit disk $\mathbf{D} = \{z \in \mathbf{C} : |z| < 1\}$, endowed with its Poincaré metric, i.e. the metric obtained by projecting the Poincaré metric of the unit disk $ds = 2|dz|/(1 - |z|^2)$ or, equivalently, the upper half plane $\mathbf{U} = \{z \in \mathbf{C} : \text{Im } z > 0\}$, with the metric $ds = |dz|/\text{Im } z$. Observe that, with this definition, every compact non-exceptional Riemann surface without boundary is open. With this metric, S is a geodesically complete Riemannian manifold with constant curvature -1 , and therefore S is a geodesic metric space. The only Riemann surfaces which are left out are the sphere, the plane, the punctured plane and the tori. It is easy to study the hyperbolicity of these particular cases.

It is well-known (see e.g. [JS, p. 227]) that

$$(3.1) \quad d_{\mathbf{D}}(0, z) = \log \frac{1 + |z|}{1 - |z|} = 2 \operatorname{Argtanh} |z|, \quad \sinh^2 \frac{d_{\mathbf{U}}(z, w)}{2} = \frac{|z - w|^2}{4 \operatorname{Im} z \operatorname{Im} w}.$$

Let S be an open non-exceptional Riemann surface with a puncture q (if $S \subset \mathbf{C}$, every isolated point in ∂S is a puncture). A *collar* in S about q is a doubly connected domain in S “bounded” both by q and a Jordan curve (called the boundary curve of the collar) orthogonal to the pencil of geodesics emanating from q .

A collar in S about q of area α will be called an α -*collar* and it will be denoted by $C_S(q, \alpha)$. A theorem of Shimizu [S] gives that for every puncture in any open non-exceptional Riemann surface, there exists an α -collar for every $0 < \alpha \leq 2$ (see also [B, Chapter 4.4]).

We say that a curve is *homotopic to a puncture* q if it is freely homotopic to $\partial C_S(q, \alpha)$ for some (and then for every) $0 < \alpha < 2$.

We have used the word *geodesic* in the sense of Definition 2.2, that is to say, as a global geodesic or a minimizing geodesic; however, we need now to deal with a special type of local geodesics: simple closed geodesics, which obviously can not be minimizing geodesics. We will continue using the word *geodesic* with the meaning of Definition 2.2, unless we are dealing with closed geodesics.

A *collar* in S about a simple closed geodesic γ is a doubly connected domain in S “bounded” by two Jordan curves (called the boundary curves of the collar) orthogonal to the pencil of geodesics emanating from γ ; such collar is equal to $\{p \in S : d_S(p, \gamma) < d\}$, for some positive constant d . The constant d is called the *width* of the collar. The Collar Lemma [R] says that there exists a collar of γ of width d , for every $0 < d \leq d_0$, where $\cosh d_0 = \coth(L_S(\gamma)/2)$ (see also [B, Chapter 4]).

We say that S is a *bordered non-exceptional Riemann surface* (or a non-exceptional Riemann surface with boundary) if it can be obtained deleting an open set V of an open non-exceptional Riemann surface R , such that:

- (1) S is connected and $d_S := d_R|_S$ (recall Definition 2.5),
- (2) any ball in R intersects at most a finite number of connected components of V ,
- (3) the boundary of S is locally Lipschitz.

Any such surface S is a bordered orientable Riemannian manifold of dimension 2 and its Riemannian metric has constant negative curvature -1 . It is not difficult to see that S is a geodesic metric space.

A *funnel* is a bordered non-exceptional Riemann surface which is topologically a cylinder and whose boundary is a simple closed geodesic. Given a positive number a , there is a unique (up to conformal mapping) funnel such that its boundary curve has length a . Every funnel is conformally equivalent, for some $\beta > 1$, to the subset $\{z \in \mathbf{C} : 1 \leq |z| < \beta\}$ of the annulus $\{z \in \mathbf{C} : 1/\beta < |z| < \beta\}$.

Every doubly connected end of an open non-exceptional Riemann surface is a puncture (if there are homotopically non-trivial curves with arbitrary small length) or a funnel (if this was not so).

A *Y-piece* is a bordered non-exceptional Riemann surface which is conformally equivalent to a sphere without three open disks and whose boundary curves are simple closed geodesics. Given three positive numbers a, b, c , there is a unique (up to conformal mapping) *Y-piece* such that their boundary

curves have lengths a, b, c (see e.g. [B, p. 109]). They are a standard tool for constructing Riemann surfaces. A clear description of these Y -pieces and their use is given in [C, Chapter X.3] and [B, Chapter 3].

A *generalized Y -piece* is a non-exceptional Riemann surface (with or without boundary) which is conformally equivalent to a sphere without n open disks and m points, with integers $n, m \geq 0$ such that $n + m = 3$, the n boundary curves are simple closed geodesics and the m deleted points are punctures. Observe that a generalized Y -piece is topologically the union of a Y -piece and m cylinders, with $0 \leq m \leq 3$.

By *the* collar of a puncture we mean the 2-collar. By *the* collar of a simple closed geodesic we mean the collar of width d_0 , where $\cosh d_0 = \coth(L_S(\gamma)/2)$. We have that two collars (corresponding to two distinct punctures, two disjoint geodesics or to one puncture and one geodesic) in S are always disjoint (see e.g. [B, p. 112]).

Although the following result is an important tool in the proof of Theorem 3.4, it is interesting by itself as well. Let us observe that it gives universal constants which depend neither on the surface S nor on the point $p \in S$, in a similar way to the Collar Lemma.

Theorem 3.1. *Let us consider an open non-exceptional Riemann surface S and $p \in S$. If in $B_S(p, r)$ there is a closed curve freely homotopic to a puncture or to a simple closed geodesic γ and $r \leq \frac{1}{2} \log 3$, then $\overline{B_S(p, r)}$ is contained in the collar of γ . Consequently, $\overline{B_S(p, r)}$ is simply or doubly connected, and $\partial B_S(p, r)$ has at most two connected components.*

Proof. We consider the ball $B_S(p, r)$ containing a closed curve freely homotopic to a puncture γ , with $r \leq \frac{1}{2} \log 3$. We also consider a universal covering map $\pi : \mathbf{U} \rightarrow S$. We can assume, without loss of generality, that $\pi(\{0 \leq \operatorname{Re} z < 1, \operatorname{Im} z > 1/2\})$ is the 2-collar of γ , and that $\pi(it) = \pi(1 + it) = p$, for some $t > 0$. There is a geodesic (except in the point p) γ_1 freely homotopic to γ , starting and finishing in p , with length less than $2r$. We consider the lift γ_2 of γ_1 to \mathbf{U} starting in it and finishing in $1 + it$. By (3.1), we have that

$$\sinh^2 r > \sinh^2 \frac{L_S(\gamma_1)}{2} = \sinh^2 \frac{d_{\mathbf{U}}(it, 1 + it)}{2} = \frac{1}{4t^2}, \quad t > \frac{1}{2 \sinh r}.$$

Since $r \leq \frac{1}{2} \log 3$, we obtain $te^{-r} > e^{-r}/(2 \sinh r) = 1/(e^{2r} - 1) \geq 1/2$. Then (see e.g. [JS, p. 227]), we have $\overline{B_{\mathbf{U}}(it, r)} = \{(\operatorname{Re} z)^2 + (\operatorname{Im} z - t \cosh r)^2 \leq t^2 \sinh^2 r\} \subset \{\operatorname{Im} z \geq te^{-r}\} \subset \{\operatorname{Im} z > 1/2\}$.

Consequently, $\overline{B_S(p, r)} \subset C_S(q, 2)$, and this fact implies that $\overline{B_S(p, r)}$ is doubly connected and $\partial B_S(p, r)$ is the union of two simple closed curves.

We consider now the ball $B_S(p, r)$ containing a closed curve freely homotopic to a simple closed geodesic γ with length $L_S(\gamma) = 2l$. We consider a universal covering map $\pi : \mathbf{U} \rightarrow S$ with $\pi(\{\operatorname{Re} z = 0\}) = \gamma$. Then $\pi(\{\rho e^{i\phi} : 1 \leq \rho < e^{2l}, |\phi - \pi/2| < \operatorname{arccosh}(\cosh d)\})$ is the collar of γ of width $d \leq d_0$, if $\cosh d_0 = \coth l$ (by the Collar Lemma). Without loss of generality, we can assume that $\pi(ie^{-i\theta}) = \pi(ie^{2l-i\theta}) = p$, for some $0 < \theta < \pi/2$. There is a geodesic (except in the point p) γ_3 , freely homotopic to γ , starting and finishing in p , with $2l \leq L_S(\gamma_3) < 2r$. We consider the lift γ_4 of γ_3 to \mathbf{U} starting in $ie^{-i\theta}$ and finishing in $ie^{2l-i\theta}$. By (3.1), we have that

$$\sinh^2 r > \sinh^2 \frac{L_S(\gamma_3)}{2} = \sinh^2 \frac{d_{\mathbf{U}}(ie^{-i\theta}, ie^{2l-i\theta})}{2}, \quad \sinh r > \frac{e^{2l} - 1}{2e^l \cos \theta} = \sinh l \sec \theta.$$

If we define $s := d_S(p, \gamma) = d_U(i, ie^{-i\theta})$, then $\cosh s = \sec \theta$ and $\sinh r > \sinh l \cosh s$.

We will prove now $s + r < d_0$; consequently $\overline{B_S(p, r)}$ is contained in the collar of γ of width d_0 , and this fact implies that $\overline{B_S(p, r)}$ is doubly connected and $\partial B_S(p, r)$ is the union of two simple closed curves; this will finish the proof of Theorem 3.1. Observe that the function $f(l) := (2 + \cosh l)/(2 - \cosh l)$ is an increasing function in $l \in [0, \frac{1}{2} \log 3]$, since $\cosh l \leq \cosh(\frac{1}{2} \log 3) = 1/\sqrt{3} < 2$; then $f(l) \geq f(0) = 3 \geq e^{2r}$ for $l < r \leq \frac{1}{2} \log 3$. Therefore, since $l < r \leq \frac{1}{2} \log 3$, we have

$$e^{2r} \leq \frac{2 + \cosh l}{2 - \cosh l} = \frac{(2 + \cosh l)(1 + \cosh l)}{(2 - \cosh l)(1 + \cosh l)} = \frac{2 + 3 \cosh l + \cosh^2 l}{2 + \cosh l - \cosh^2 l},$$

and then $e^{2r}(\sinh^2 l - \cosh l - 1) + 2 + 3 \cosh l + \cosh^2 l \geq 0$. Consequently,

$$\begin{aligned} e^{2r} \sinh^2 r - e^{2r} \sinh^2 l &\leq \cosh^2 l + 2 \cosh l - (e^{2r} - 1) \cosh l + 1 + e^{2r} \sinh^2 r - (e^{2r} - 1) \\ e^{2r} \sinh^2 r - e^{2r} \sinh^2 l &\leq \cosh^2 l + 2(1 - e^r \sinh r) \cosh l + 1 + e^{2r} \sinh^2 r - 2e^r \sinh r \\ e^{2r}(\sinh^2 r - \sinh^2 l) &\leq (\cosh l + 1 - e^r \sinh r)^2 \\ e^r \sqrt{\sinh^2 r - \sinh^2 l} &\leq \cosh l + 1 - e^r \sinh r, \end{aligned}$$

since $\cosh l + 1 > e^r \sinh r$ (in fact, $r \leq \frac{1}{2} \log 3$ gives $e^{2r} - 1 \leq 2 < 2 \cosh l + 2$). Then we have

$$\begin{aligned} e^r &\leq \frac{\cosh l + 1}{\sinh r + \sqrt{\sinh^2 r - \sinh^2 l}}, \\ r &\leq \log \frac{\cosh l + 1}{\sinh r + \sqrt{\sinh^2 r - \sinh^2 l}} = \log \frac{\frac{\cosh l}{\sinh l} + \sqrt{\frac{\cosh^2 l}{\sinh^2 l} - 1}}{\frac{\sinh r}{\sinh l} + \sqrt{\frac{\sinh^2 r}{\sinh^2 l} - 1}} \\ &= \operatorname{Argcosh} \frac{\cosh l}{\sinh l} - \operatorname{Argcosh} \frac{\sinh r}{\sinh l} = d_0 - \operatorname{Argcosh} \frac{\sinh r}{\sinh l}. \end{aligned}$$

Consequently, since $\sinh r > \sinh l \cosh s$, we obtain

$$s + r \leq d_0 + s - \operatorname{Argcosh} \frac{\sinh r}{\sinh l} < d_0.$$

Hence, $\overline{B_S(p, r)}$ is contained in the collar of γ . \square

The hyperbolicity constants of some simple Riemann surfaces can be uniformly bounded by means of the two following propositions. These propositions play a fundamental role in the proof of Theorem 3.2.

Proposition 3.1. *Let S be a simply or doubly connected bordered non-exceptional Riemann surface, such that $L_S(\partial S) \leq a$. Then S is δ -hyperbolic, where δ is a constant which only depends on a .*

Remark. As usual, we see a puncture as a geodesic of zero length.

Proof. It is well known that S is isometric to a bordered surface S_1 contained in R , where R is the unit disk \mathbf{D} , the punctured disk \mathbf{D}^* or some annulus $N_\varepsilon := \{z \in \mathbf{C} : \varepsilon < |z| < 1\}$, for $0 < \varepsilon < 1$; then R is the union of S_1 and at most two other bordered surfaces. Without loss of generality, we can assume $S_1 = S$. Observe that the diameter in R of each connected component of ∂S is less or equal than a .

If $R = \mathbf{D}$ or $R = \mathbf{D}^*$, Theorem B (with $A_n = \emptyset$) gives that S is k_4 -hyperbolic, since \mathbf{D} and \mathbf{D}^* are hyperbolic (see [RT1, Theorem 3.3]), where k_4 is a constant which only depends on a (this is the case if S is simply connected).

If $R = N_\varepsilon$ and γ is the simple closed geodesic in N_ε , we have that $L_{N_\varepsilon}(\gamma) \leq L_S(\partial S) \leq a$. Proposition 3.1 in [RT1] gives that N_ε is k_5 -hyperbolic, where k_5 is a constant which only depends on a . By Theorem B (with $A_n = \emptyset$), S is k'_4 -hyperbolic, since N_ε is k_5 -hyperbolic, where k'_4 is a constant which only depends on k_5 and a .

The proof finishes taking $\delta := \max\{k_4, k'_4\}$. \square

We also need the following result.

Theorem D. ([RT2, Theorem 3.6]) *Let us consider a non-exceptional Riemann surface S (with or without boundary) without genus. If there is a decomposition of S in a union of funnels $\{F_m\}_{m \in M}$ and generalized Y -pieces $\{Y_n\}_{n \in N}$ with $L_S(\gamma) \leq a$ for at least two simple closed geodesic $\gamma \subset \partial Y_n$ for every $n \in N$, then S is δ -hyperbolic, where δ is a constant which only depends on a .*

We can obtain a similar result to Proposition 3.1 for triply connected surfaces, using Theorem D.

Proposition 3.2. *Let S be a triply connected bordered non-exceptional Riemann surface, such that ∂S is the union of two simple closed curves verifying $L_S(\partial S) \leq a$. Then S is δ -hyperbolic, where δ is a constant which only depends on a .*

Proof. It is well known that S is isometric to a bordered surface S_1 contained in an open non-exceptional Riemann surface R , where R is the unit disk, the punctured disk, an annulus or the union of a generalized Y -piece Y_0 and at most 3 funnels. Without loss of generality, we can assume $S_1 = S$.

If R is the unit disk, the punctured disk or an annulus, we proceed as in the proof of Proposition 3.1. If this was not so, R is the union of S and two bordered surfaces. Let us observe that the diameter in R of each connected component of ∂S is less or equal than a . If g_1 and g_2 are the simple closed curves in ∂S , we denote by γ_i the simple closed geodesic in R freely homotopic to g_i ($i = 1, 2$). As $L_R(\gamma_i) \leq L_S(g_i) \leq a$, Theorem D guarantees that R is k -hyperbolic, where k is a constant which only depends on a . By Theorem B (with $A_n = \emptyset$), S is δ -hyperbolic, where δ is a constant which only depends on a . \square

The following result will be an important tool in order to prove our next theorems.

Lemma C. ([APR, Lemma 3.1]) *Let us consider an open non-exceptional Riemann surface S , a closed non-empty subset C of S , and a positive number ε . If $S^* := S \setminus C$, then we have that $1 < L_{S^*}(\gamma)/L_S(\gamma) < \coth(\varepsilon/2)$, for every curve $\gamma \subset S$ with finite length in S such that $d_S(\gamma, C) \geq \varepsilon$.*

We need the following definitions in order to state one of our main theorems.

Definition 3.1. A *normal neighborhood* of a subset F of a Riemann surface is a compact bordered Riemann surface V such that $F \subset V$, V has connection order n (with $n \in \{1, 2\}$) and ∂V is the union of n closed curves.

A set $E = \cup_n E_n$ in an open non-exceptional Riemann surface S , with $\{E_n\}_n$ compact simply connected sets, is called (r, s) -*uniformly separated* in S if there exist normal neighborhoods V_n of E_n

such that $d_S(\partial V_n, E_n) \geq r$, $L_S(\partial V_n) \leq s$ for every n , and $d_S(V_n, V_m) \geq r$ for every $n \neq m$ (if ∂V_n is not connected, by $L_S(\partial V_n)$ we mean the sum of the lengths of the connected components of ∂V_n).

Definition 3.2. Let S be an open non-exceptional Riemann surface, $E = \cup_n E_n$ a (r, s) -uniformly separated set in S and $S^* := S \setminus E$. For each choice of $\{V_n\}_n$ we define

$$\begin{aligned} D_S &= D_S(\{V_n\}_n) := \sup_n \{d_S|_{V_n}(\eta_n^1, \eta_n^2) : \eta_n^1, \eta_n^2 \text{ are the connected components of } \partial V_n \\ &\quad \text{and } S \setminus \eta_n^j \text{ is connected for } j = 1, 2\}, \\ D_{S^*} &= D_{S^*}(\{V_n\}_n) := \sup_n \{d_{S^*}|_{V_n \setminus E_n}(\eta_n^1, \eta_n^2) : \eta_n^1, \eta_n^2 \text{ are the connected components of } \partial V_n \\ &\quad \text{and } S \setminus \eta_n^j \text{ is connected for } j = 1, 2\}. \end{aligned}$$

Lemma 3.1. *Let S be an open non-exceptional Riemann surface and $E = \cup_n E_n$ a (r, s) -uniformly separated set in S . Let us assume that we can choose the sets $\{V_n\}_n$ such that $D_S(\{V_n\}_n) = \infty$ (respectively $D_{S^*}(\{V_n\}_n) = \infty$). Then S (respectively S^*) is not hyperbolic.*

Proof. For each positive integer k , we can choose V_{n_k} such that ∂V_{n_k} has two connected components η_k^1, η_k^2 , with $S \setminus \eta_k^i$ connected and $d_S|_{V_{n_k}}(\eta_k^1, \eta_k^2) \geq 4k$.

Since $L_S(\eta_k^1 \cup \eta_k^2) \leq s$, Theorem 2.2 gives that there exists a $(1, 2s)$ -quasigeodesic triangle which is δ -thin with $\delta \geq k$. Then Lemma B gives that S is not hyperbolic.

We have a similar result for S^* , since $L_{S^*}(\eta_k^1 \cup \eta_k^2) \leq L_S(\eta_k^1 \cup \eta_k^2) \coth(r/2) \leq s \coth(r/2)$ (condition $d_S(\eta_k^i, E) \geq r$ allows to apply Lemma C). \square

Since $D_S(\{V_n\}_n) \leq D_{S^*}(\{V_n\}_n)$ by Lemma C, we deduce the following result.

Corollary 3.1. *Let S be an open non-exceptional Riemann surface and $E = \cup_n E_n$ a (r, s) -uniformly separated set in S . Let us assume that we can choose the sets $\{V_n\}_n$ such that $D_S(\{V_n\}_n) = \infty$. Then S and S^* are not hyperbolic.*

The next result allows, in many cases, to forget punctures and funnels in order to study the hyperbolicity of a Riemann surface; this fact can be a significant simplification in the topology of the surface, and therefore makes easier the study of its hyperbolicity. Recall that to delete each E_n which is (respectively, is not) an isolated point gives a puncture (respectively, a funnel) in S^* .

Let us remark that we consider simply connected sets E_n since we are interested in punctures and funnels. However, the condition “ E_n is simply connected” is essentially equivalent to “ E_n is connected”: we can assume that there is no non-trivial simple closed curve σ in E_n , since it is rather artificial to consider S^* as a subset of a surface S with more topological obstacles than S^* .

Theorem 3.2. *Let S be an open non-exceptional Riemann surface and $E = \cup_n E_n$ a (r, s) -uniformly separated set in S . Then, $S^* := S \setminus E$ is δ^* -hyperbolic if and only if S is δ -hyperbolic and $D_{S^*}(\{V_n\}_n)$ is finite. Furthermore, if $D_{S^*}(\{V_n\}_n)$ is finite, δ^* (respectively δ) is a universal constant which only depends on $r, s, D_{S^*}(\{V_n\}_n)$ and δ (respectively δ^*).*

Remark. Recall that $d_{S^*} \neq d_S|_{S^*}$, since (S^*, d_{S^*}) is a complete Riemannian manifold (the points of E are at infinite d_{S^*} -distance of the points of S^*). This fact also implies that (S^*, d_{S^*}) is geodesically complete (it is an open non-exceptional Riemann surface).

Proof. If $D_{S^*}(\{V_n\}_n) = \infty$, Lemma 3.1 gives that S^* is not hyperbolic. We see now that if $D_{S^*}(\{V_n\}_n) < \infty$, S^* is hyperbolic if and only if S is hyperbolic. This fact finishes the proof.

Theorem C is an important tool in this proof. In order to apply it, we need to construct bordered Riemann surfaces U_n with better properties than V_n . If ∂V_n is connected or if ∂V_n has two connected components η_n^1, η_n^2 , with $d_{V_n}(\eta_n^1, \eta_n^2) \geq r/2$, we define $U_n := V_n$. If ∂V_n has connected components η_n^1, η_n^2 , with $d_{V_n}(\eta_n^1, \eta_n^2) < r/2$, we define U_n in the following way: we choose two disjoint Lipschitz curves s_n^1, s_n^2 in V_n joining η_n^1 and η_n^2 , with $L_S(s_n^j) < r/2$; since V_n is a doubly connected set, there exists a unique simply connected compact bordered Riemann surface $U_n \subset V_n$ with $E_n \subset U_n$ and $s_n^1, s_n^2 \subset \partial U_n \subset \partial V_n \cup s_n^1 \cup s_n^2$.

It is clear that U_n is a normal neighborhood of E_n . Since $L_S(s_n^j) < r/2$ and $U_n \subset V_n$, we have that $d_S(\partial U_n, E_n) \geq r/2 =: r_0$, $L_S(\partial U_n) \leq s + r =: s_0$ for every n , and $d_S(U_n, U_m) \geq d_S(V_n, V_m) \geq r \geq r_0$ for every $n \neq m$. Then E is (r_0, s_0) -uniformly separated in S if we choose $\{U_n\}_n$ as normal neighborhoods. We also have $D'_{S^*} := D_{S^*}(\{U_n\}_n) \leq D_{S^*}(\{V_n\}_n)$, and if ∂U_n has two connected components σ_n^1, σ_n^2 , then $d_{U_n}(\sigma_n^1, \sigma_n^2) \geq r/2 = r_0$.

Let us denote by K the set of indices of $\{U_n\}_n$. For each $n \in K$, let us define $X_n := U_n$ and $X_n^* := U_n \setminus E_n$.

Let us consider the connected components $\{X_n\}_{n \in J}$ of $S_0 := S \setminus \bigcup_{n \in K} \text{int } U_n$. If we define $X_n^* := X_n$ for $n \in J$, then $S = \bigcup_{n \in \Lambda} X_n$ and $S^* = \bigcup_{n \in \Lambda} X_n^*$, with $\Lambda := K \cup J$. We have that each X_n (with the restricted metric of S) and X_n^* (with the restricted metric of S^*) are bordered non-exceptional Riemann surfaces, for any $n \in \Lambda$; hence they are geodesic metric spaces.

We define the set G_0 as the set of indices $n \in K$ such that ∂U_n has two connected components σ_n^1, σ_n^2 , and $d_S|_{S_0}(\sigma_n^1, \sigma_n^2) < r_0$.

In order to apply Theorem C, let us prove that S and S^* (and S^* and S) have comparable decompositions, given by $\{X_n\}_{n \in \Lambda}$ and $\{X_n^*\}_{n \in \Lambda}$:

(a) We have $X_n \cap X_m = X_n^* \cap X_m^* =: \bigcup_{i \in I_{nm}} \eta_{nm}^i$, where we define η_{nm}^i as follows: η_{nm}^i is a simple closed curve if $n, m \notin G_0$ (then I_{nm} has at most two elements); if $n \in G_0$ or $m \in G_0$, $\eta_{nm} := X_n \cap X_m = X_n^* \cap X_m^*$ (then I_{nm} has at most one element, although η_{nm} can have two connected components: we do not have any hypothesis about the connection of η_{nm}^i in definitions 2.7 and 2.8).

Any geodesic segment in S meets at most a finite number of η_{nm}^i 's, since $d_S(U_a, U_b) \geq r$ for any $a, b \in K$ with $a \neq b$. The same result is true in S^* .

(b) Lemma C guarantees that $\text{diam}_{X_n}(\eta_{nm}^i) \leq \text{diam}_{X_n^*}(\eta_{nm}^i) \leq s_0 \coth(r_0/2)$, if $n, m \notin G_0$; we also have $\text{diam}_{X_n}(\eta_{nm}) \leq \text{diam}_{X_n^*}(\eta_{nm}) \leq L_{X_n}(\partial U_n) \coth(r_0/2) + D'_{S^*} \leq s_0 \coth(r_0/2) + D'_{S^*}$, if $n \in G_0$ or $m \in G_0$. In fact, if $n \in K$, we have $\sum_{m,i} \text{diam}_{X_n}(\eta_{nm}^i) \leq \sum_{m,i} \text{diam}_{X_n^*}(\eta_{nm}^i) \leq L_{X_n}(\partial U_n) \coth(r_0/2) + D'_{S^*} \leq s_0 \coth(r_0/2) + D'_{S^*}$.

(c) We can split Λ into $F_1 \cup F_2 \cup G$ with $G := G_0 \cup (K \setminus L)$, $F_1 := L \setminus G_0$ and $F_2 := J$, where L is the set of indices $n \in K$ such that $S \setminus \sigma_n^j$ is connected for some connected component σ_n^j of ∂U_n (let us observe that $S \setminus \sigma_n^1$ is connected if and only if $S \setminus \sigma_n^2$ is connected, since σ_n^1 and σ_n^2 are freely homotopic). Then:

(c1) If $n \in G$, X_n (and X_n^*) is a $(s_0 \coth(r_0/2) + D'_{S^*}, 0, s_0 \coth(r_0/2) + D'_{S^*})$ -tree-piece, if we

choose $A_n = \emptyset$: if $n \in K \setminus L$, each connected component of ∂U_n disconnects S , and consequently, $\sharp I_{nm} \leq 1$; if $n \in G_0$, there is just one $\eta_{nm} = \sigma_n^1 \cup \sigma_n^2$ and hence $S \setminus \eta_{nm} = S \setminus \{\sigma_n^1 \cup \sigma_n^2\}$ is not connected.

(c2) Let us consider $n \in F_2 = J$; if $m \neq k$, $d_{X_n^*}(\eta_{nm}^i, \eta_{nk}^j) \geq d_{X_n}(\eta_{nm}^i, \eta_{nk}^j) \geq r_0$, since $d_S(U_m, U_k) \geq r_0$; if $m \in F_1$, $d_{X_n^*}(\eta_{nm}^1, \eta_{nm}^2) \geq d_{X_n}(\eta_{nm}^1, \eta_{nm}^2) = d_{X_n}(\sigma_m^1, \sigma_m^2) \geq r_0$, since $m \notin G_0$; if $m \in G$, there is just one η_{nm} . If $n \in F_1 = L \setminus G_0$, we have $d_{X_n^*}(\eta_{nm}^i, \eta_{nk}^j) \geq d_{X_n}(\eta_{nm}^i, \eta_{nk}^j) = d_{X_n}(\sigma_n^1, \sigma_n^2) \geq r_0$. (See the remark after Definition 2.8.)

(c3) If $n \in F_1 = L \setminus G_0$, we consider geodesics γ_{mnk}^{ij} and h_{mnk}^{ij} in X_n and X_n^* respectively, joining $\eta_{nm}^i = \sigma_n^1$ and $\eta_{nk}^j = \sigma_n^2$, with $L_S(\gamma_{mnk}^{ij}) = d_{X_n}(\sigma_n^1, \sigma_n^2) \geq r_0$ and $L_{S^*}(h_{mnk}^{ij}) = d_{X_n^*}(\sigma_n^1, \sigma_n^2) \leq D'_{S^*}$; if we define f_{mnk}^{ij} as the dilatation between γ_{mnk}^{ij} and h_{mnk}^{ij} , it is a $(D'_{S^*}/r_0, 0)$ -quasiisometry. We do not need to check the last condition in (c3) since ∂U_n has just two connected components.

(c4) If $n \in F_2 = J$, we have that the identity $i_n : X_n \longrightarrow X_n^*$ is a $(\coth(r_0/2), 0)$ -quasiisometry by Lemma C.

Then S and S^* (and S^* and S^*) have comparable decompositions, given by $\{X_n\}_{n \in \Lambda}$ and $\{X_n^*\}_{n \in \Lambda}$.

For any $n \in K$, we have that $X_n = U_n$ is a compact bordered non-exceptional Riemann surface with connection order $n \leq 2$, such that ∂U_n is the union of n closed curves. Since $L_S(\partial U_n) \leq s_0$, Proposition 3.1 guarantees that X_n is k_6 -hyperbolic, with a constant k_6 which only depends on s_0 .

For any $n \in K$, we have that $X_n^* = U_n \setminus E_n$ is a compact bordered non-exceptional Riemann surface with connection order $n + 1 \leq 3$, such that ∂U_n is the union of n closed curves. Since $L_{S^*}(\partial U_n) \leq L_S(\partial U_n) \coth(r_0/2) \leq s_0 \coth(r_0/2)$ by Lemma C, propositions 3.1 and 3.2 guarantee that X_n^* is k_6^* -hyperbolic, with a constant k_6^* which only depends on r_0 and s_0 .

Let us observe that $\Lambda \setminus F_2 = K$. Consequently, Theorem C gives that if S^* is δ^* -hyperbolic, then S is δ -hyperbolic, where δ only depends on r, s, D_{S^*} and δ^* , and that if S is δ -hyperbolic, then S^* is δ^* -hyperbolic, where δ^* only depends on r, s, D_{S^*} and δ . \square

Theorem 3.2 has the following direct consequence.

Corollary 3.2. *Let S be an open non-exceptional Riemann surface and $E = \cup_n E_n$ a (r, s) -uniformly separated set in S . Let us assume also that we can choose the sets $\{V_n\}_n$ such that every connected component of each ∂V_n disconnects S . Then, S is δ -hyperbolic if and only if $S^* := S \setminus E$ is δ^* -hyperbolic. Furthermore, δ^* (respectively δ) is a universal constant which only depends on r, s and δ (respectively δ^*).*

Next we introduce a concept which will be used in the theorems below.

Definition 3.3. If c is a positive constant, we say that a non-exceptional Riemann surface S (with or without boundary) has c -wide genus if every homotopically non-trivial simple closed curve $\gamma \subset S$ such that $S \setminus \gamma$ is connected, verifies $L_S(\gamma) \geq c$. We say that S has *narrow genus* if there is not $c > 0$ such that S has c -wide genus.

Observe that if S is open, it has c -wide genus if and only if every simple closed geodesic $\gamma \subset S$ such that $S \setminus \gamma$ is connected, verifies $L_S(\gamma) \geq c$.

Notice that any plane domain has c -wide genus for every c , and that any Riemann surface with finite genus has c -wide genus for some c .

We will need the following general criteria which guarantees that many surfaces are not hyperbolic. It is used in the proofs of Theorem 3.4, and [RT2, Theorem 3.8].

Theorem 3.3. *Any non-exceptional Riemann surface (with or without boundary) with narrow genus is not hyperbolic.*

Proof. Let us consider first an open non-exceptional Riemann surface S with narrow genus. We choose a sequence of simple closed geodesics $\{\gamma_n\}_n$ in S with $S \setminus \gamma_n$ connected and $\lim_{n \rightarrow \infty} L_S(\gamma_n) = 0$.

The Collar Lemma [R] says that there exists a collar of γ_n of width d , for every $0 < d \leq d_n$, where $\cosh d_n = \coth(L_S(\gamma_n)/2)$.

We define the bordered Riemann surfaces S_2^n as the collar of γ_n of width $d_n/2$, and $S_1^n := \overline{S \setminus S_2^n}$, which is connected since $S \setminus \gamma_n$ is connected. We have that $\partial S_1^n = \partial S_2^n = S_1^n \cap S_2^n = \eta_1^n \cup \eta_2^n$, with

$$L_S(\eta_i^n) = L_S(\gamma_n) \cosh(d_n/2) = L_S(\gamma_n) \sqrt{\frac{\cosh d_n + 1}{2}} = L_S(\gamma_n) \sqrt{\frac{\coth(L_S(\gamma_n)/2) + 1}{2}}.$$

Since S_2^n is the collar of γ_n of width $d_n/2$, we have that $d_S(\eta_1^n, \eta_2^n) = d_{S_2^n}(\eta_1^n, \eta_2^n) = d_n$. By Theorem 2.1, if S is δ -thin, then $\delta \geq d_n/2 - L_S(\eta_i^n)/2$. Since $\lim_{n \rightarrow \infty} d_n = \infty$ and $\lim_{n \rightarrow \infty} L_S(\eta_i^n) = 0$, we have that S is not hyperbolic.

If S has boundary, it is contained in an open non-exceptional Riemann surface R . Let us choose simple closed curves $\{g_n\}_n$ in S with $S \setminus g_n$ connected and $\lim_{n \rightarrow \infty} L_S(g_n) = 0$. Let us consider the simple closed geodesic γ_n in R freely homotopic to g_n ; we have that $R \setminus \gamma_n$ is connected and $\lim_{n \rightarrow \infty} L_R(\gamma_n) = 0$. Each geodesic γ_n has in R a collar of width d , for every $0 < d \leq d_n$, with $\cosh d_n = \coth(L_S(\gamma_n)/2)$.

We define the bordered Riemann surfaces R_2^n as the collar of γ_n in R of width $d_n/2$, S_2^n as a connected component of $S \cap R_2^n$ such that $S \setminus S_2^n$ is connected, and $S_1^n := \overline{S \setminus S_2^n}$. We have that $\partial S_1^n = \partial S_2^n = S_1^n \cap S_2^n = \eta_1^n \cup \eta_2^n$, with $L_S(\eta_i^n) \leq L_R(\gamma_n) \sqrt{(\coth(L_R(\gamma_n)/2) + 1)/2}$, and $d_{S_2^n}(\eta_1^n, \eta_2^n) \geq d_n$. Since $\lim_{n \rightarrow \infty} d_{S_2^n}(\eta_1^n, \eta_2^n) = \infty$ and $\lim_{n \rightarrow \infty} L_S(\eta_i^n) = 0$, we have that S is not hyperbolic, by Theorem 2.2 and Lemma B. \square

If E_n is a single point for every n , theorems 3.1, 3.2 and 3.3 allow to prove a result with a statement much simpler than Theorem 3.2; in fact, S^* is hyperbolic if and only if S is hyperbolic (we do not need to consider D_{S^*}). This theorem is also an improvement of [RT1, Theorem 3.3], in the direction of weakening the hypothesis on the set E (see [RT1]). We need a definition.

Definition 3.4. A set E in an open non-exceptional Riemann surface S is called *r-uniformly separated* if the balls $\{B_S(p, r)\}_{p \in E}$ are pairwise disjoint.

The r -uniformly separated sets play a central role in the study of hyperbolic isoperimetric inequalities in open Riemann surfaces (see [APR, Theorem 1] and [FR1, Theorems 3 and 4]). There are interesting relations of the hyperbolic isoperimetric inequality with other conformal invariants of a Riemann surface (see e.g. [APR], [C, p. 95], [FR1], [Su, p. 333]).

Theorem 3.4. *Let S be an open non-exceptional Riemann surface and E a r -uniformly separated set in S . Then, $S^* := S \setminus E$ is δ^* -hyperbolic if and only if S is δ -hyperbolic. Furthermore, δ^**

(respectively δ) only depends on c, r and δ (respectively δ^*), where c is the best constant such that S has c -wide genus.

The conclusion of Theorem 3.4 is not true without the hypothesis about E , even for plane domains: it is sufficient to consider $S := \mathbf{C} \setminus \{0, 1\}$ (which is hyperbolic by Theorem D) and $S^* := \mathbf{C} \setminus \mathbf{Z}^2$ (which is not hyperbolic since it has an isometry group isomorphic to \mathbf{Z}^2).

Proof. We assume first that S has c -wide genus, for some c . For each p , the set of r 's such that $\partial B(p, r)$ is not the union of simple closed curves (that is to say, $\overline{B_S(p, r)}$ is not a bordered Riemann surface) is at most countable. Since E is at most countable, the set of r 's such that $\overline{B_S(p, r)}$ is not a bordered Riemann surface for some $p \in E$, is at most countable. Let us consider $r_0 < \frac{1}{2} \min\{c, r, \log 3\}$, such that $\overline{B_S(p, r_0)}$ is a bordered Riemann surface for every $p \in E$.

We see now that E is a $(r_0, 2\pi \sinh r_0)$ -uniformly separated set in S , with normal neighborhoods $V_p := \overline{B_S(p, r_0)}$. We have for any $p \in E$ that $\overline{B_S(p, r_0)}$ is simply or doubly connected by Theorem 3.1. Furthermore, each connected component of $\partial B_S(p, r_0)$ disconnects S : This is clear if $B_S(p, r_0)$ is simply connected. If $B_S(p, r_0)$ is not simply connected, then there exists a non-trivial simple closed curve g in $B_S(p, r_0)$ with length less than $2r_0 < c$, and therefore g disconnects S ; we have the result since every non-trivial simple closed curve in $B_S(p, r_0)$ is freely homotopic to g by Theorem 3.1. We also have that $d_S(p, \partial B_S(p, r_0)) = r_0$ and $L_S(\partial B_S(p, r_0)) \leq 2\pi \sinh r_0$ for every $p \in E$, and $d_S(B_S(p, r_0), B_S(q, r_0)) \geq r > r_0$, for every $p \neq q$.

Hence E is a $(r_0, 2\pi \sinh r_0)$ -uniformly separated set in S , and the result follows from Corollary 3.2.

If S has narrow genus, then Theorem 3.3 guarantees that S is not hyperbolic. The same reasoning as above taking $r_1 < \frac{1}{2} \min\{r, \log 3\}$ shows that E is a $(r_1, 2\pi \sinh r_1)$ -uniformly separated set in S (the dependence of r_0 on c is just used to prove that each connected component of $\partial B_S(p, r_0)$ disconnects S). Consequently, Theorem 3.2 allows to deduce that S^* is not hyperbolic. \square

If we consider $S^* := S \setminus \{p_1, p_2\}$, where S is an open Riemann surface and $p_1, p_2 \in S$, there are several conformal invariants of S^* (e.g. the exponent of convergence and the isoperimetric constant) which degenerate when p_2 tends to p_1 . We have the following surprising consequence of Theorem 3.4 about stability of hyperbolicity.

Corollary 3.3. *Let S be a δ -hyperbolic open non-exceptional Riemann surface with c -wide genus. Then, for each natural number n there exists a constant δ_n , which only depends on δ, c and n , such that $S \setminus \{p_1, \dots, p_n\}$ is δ_n -hyperbolic, for any $p_1, \dots, p_n \in S$.*

Proof. We prove the theorem by induction on n . Theorem 3.4 gives the result for $n = 1$ ($E = \{p_1\}$ is r -uniformly separated for any r). Let us assume that the result is true for $n - 1$; then $S^* := S \setminus \{p_1, \dots, p_{n-1}\}$ is δ_{n-1} -hyperbolic, for any $p_1, \dots, p_{n-1} \in S$.

Observe that S^* has c -wide genus, since $d_{S \setminus F} \geq d_S$ for any closed set $F \subset S$. Then Theorem 3.4 gives that $S^* \setminus \{p_n\}$ is δ_n -hyperbolic, where δ_n is a constant which only depends on δ_{n-1} and c ($E = \{p_n\}$ is r -uniformly separated for any r). \square

Now we give a simple condition which implies $D_{S^*} = \infty$, just in terms of distances in S .

Definition 3.5. Let S be an open non-exceptional Riemann surface and $E = \cup_n E_n$ a (r, s) -uniformly separated set in S . For each fixed choice of $\{V_n\}_n$ we denote by L the set of indices n such that ∂V_n has some connected component η_n with $S \setminus \eta_n$ connected. If $n \in L$, let us denote by $C(E_n)$ the set of curves γ joining E_n with itself, such that in $E_n \cup \gamma$ there exists a curve σ with $S \setminus \sigma$ connected. We define

$$C_S(\{V_n\}_n) := \inf \{L_S(\gamma) : \gamma \in C(E_n) \text{ for some } n\}.$$

Proposition 3.3. Let S be an open non-exceptional Riemann surface and $E = \cup_n E_n$ a (r, s) -uniformly separated set in S . If for some choice of the sets $\{V_n\}_n$, we have $C_S(\{V_n\}_n) = 0$, then $D_{S^*}(\{V_n\}_n) = \infty$.

Proof. We can choose n_k and a geodesic γ_k which has minimal length in $C(E_{n_k})$, with $L_S(\gamma_k) = 4\varepsilon_k < r$ and $\lim_{k \rightarrow \infty} \varepsilon_k = 0$. If we consider the universal covering, we see that any curve joining the two connected components of ∂V_{n_k} in S^* is longer or equal lengthed than the shortest curve g_k in $\mathbf{D} \setminus \{i \tanh \varepsilon_k, -i \tanh \varepsilon_k\}$ joining $\{|z| = \tanh(r/4)\}$ with itself and intersecting the segment joining $i \tanh \varepsilon_k$ and $-i \tanh \varepsilon_k$ (let us observe that $d_{\mathbf{D}}(-i \tanh \varepsilon_k, i \tanh \varepsilon_k) = 2d_{\mathbf{D}}(0, i \tanh \varepsilon_k) = 4\varepsilon_k$, by (3.1)). It is not difficult to see that g_k is the interval $[-\tanh(r/4), \tanh(r/4)]$. Then $D_{S^*}(\{V_n\}_n) \geq \sup_k L_{\mathbf{D} \setminus \{i \tanh \varepsilon_k, -i \tanh \varepsilon_k\}}([-\tanh(r/4), \tanh(r/4)])$. We denote by D_t the disk with center 0 and Euclidean radius t . Since

$$\begin{aligned} \lim_{k \rightarrow \infty} L_{\mathbf{D} \setminus \{i \tanh \varepsilon_k, -i \tanh \varepsilon_k\}}([-\tanh(r/4), \tanh(r/4)]) \\ = \lim_{k \rightarrow \infty} L_{D_{\coth \varepsilon_k} \setminus \{i, -i\}}([-\tanh(r/4) \coth \varepsilon_k, \tanh(r/4) \coth \varepsilon_k]) \\ = L_{\mathbf{C} \setminus \{i, -i\}}((-\infty, \infty)) = \infty, \end{aligned}$$

we have that $D_{S^*}(\{V_n\}_n) = \infty$. \square

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